Extension of the Lee-Yang Circle Theorem

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(Received 7 December 1970)

We study the zeros of the partition function of a classical spin system and compute a region of the complex fugacity plane where they necessarily lie. We recover the Lee-Yang circle theorem as a special example; we also find that a spin system with finite-range interaction has no phase transition at high temperature.

Asano’s recent results have revived interest in the celebrated Lee-Yang circle theorem, by giving it a more conceptual proof. Here an extension of the circle theorem to noncircular regions is proved and applied to problems in statistical mechanics.

(1) Statement of results. Let \( P \) be a complex polynomial in several variables, which is of degree 1 with respect to each, i.e., \( \Lambda \) is a finite set and

\[
P(\mathbf{z}_\Lambda) = \sum_{\mathbf{z}_\Lambda} c_{\mathbf{z}_\Lambda} \mathbf{z}_\Lambda^x,
\]

where \( \mathbf{z}_\Lambda = (z_k)_{k \in \Lambda} \), and \( z^x = \prod_{x \in x \in \Lambda} z_x^x \).

Theorem: Let \( (\Lambda, \alpha) \) be a finite covering of \( \Lambda \), and for every \( x \in \Lambda \), let \( M_{\alpha x} \) be a closed subset of the complex plane \( C \) such that \( 0 \notin M_{\alpha x} \). For each \( \alpha \) we assume that the polynomial

\[
P_x(\mathbf{z}_\Lambda) = \sum_{\mathbf{z}_\Lambda} c_{\mathbf{z}_\Lambda} \mathbf{z}_\Lambda^x
\]

does not vanish when \( z_x \in -\prod_{x \in x \in \Lambda} (-M_{\alpha x}) \), all \( x \in \Lambda \).

Then the polynomial

\[
P(\mathbf{z}_\Lambda) = \sum_{\mathbf{z}_\Lambda \in \Lambda} z^x \prod_{x \in x \in \Lambda} c_{\mathbf{z}_\Lambda(\Lambda, \alpha)} x
\]

does not vanish when \( z_x \in -\prod_{x \in x \in \Lambda} (-M_{\alpha x}) \), all \( x \in \Lambda \).

The proof is given in Section 2. This result extends a theorem by Lee and Yang, and can be used in the same way to obtain regions free of zeros for polynomials in one variable. More precisely, let \( \Lambda_0 \) be the two-point subsets of \( \Lambda \): \( \Lambda_0 = \{x, y\} \) and \( c_{\alpha x} = a_{xy} \) when \( X = \{x\} \) or \( X = \{y\} \), \( c_{\alpha x} = 1 \) when \( X = \emptyset \) or \( X = \{x, y\} \). For real \( a_{xy} \) and

\[-1 \leq a_{xy} \leq 1 \]

we may take \( M_{\alpha x} = \{z \in C : |z| \geq 1\} \); hence

\[
Q(\xi) = \sum_{\mathbf{z}_\Lambda} \xi^{\mathbf{z}_\Lambda \cdot \mathbf{z}_\Lambda} \prod_{x \in x \in \Lambda} a_{xy}
\]

does not vanish when \( |\xi| < 1 \). By symmetry \( Q(\xi) \) does not vanish when \( |\xi| > 1 \), hence the zeros of \( Q \) have absolute value 1; this is the Lee-Yang circle theorem.

Let \( \Phi \) be a real function on \( (Z_m)^v \) (v-tuples of integers mod \( m \), "periodic lattice") with \( \Phi(x) = \Phi(-x) \), and take \( \Lambda = (Z_m)^v \). Let again the \( \Lambda_0 \) be the two-point subsets of \( \Lambda \); \( \Lambda_0 = \{x, y\} \), and write \( c_{\alpha x} = \exp(-\beta\Phi(x-y)) \) when \( X = \{x, y\} \) and \( c_{\alpha x} = 1 \) when \( X = \emptyset \), \( \{x\} \), or \( \{y\} \). We may then take

\[
M_{\alpha x} = \Lambda \delta_{xy}
\]

where

\[
\Lambda \delta_{xy} = \{z \in C : |z+1| \leq (1-e^{-\beta\Phi(x-y)})^{1/2}\}
\]

for \( \Phi(x-y) \leq 0 \),

\[
\Lambda \delta_{xy} = \{z \in C : |ze^{-\beta\Phi(x-y)} + 1| \leq (1-e^{-\beta\Phi(x-y)})^{1/2}\}
\]

for \( \Phi(x-y) \geq 0 \),

and we find that

\[
Q(\xi) = \sum_{\mathbf{z}_\Lambda} \xi^{\mathbf{z}_\Lambda \cdot \mathbf{z}_\Lambda} \exp(-\beta \sum_{x \in x \in \Lambda} \Phi(x-y))
\]

can vanish only when

\[
\xi e^{\Gamma \beta} = \prod_{y \in Z_m} (-\Lambda_{0y} \beta)
\]

The region \( \Gamma \beta \) is sketched for small and large values of \( \beta \) in Fig. 1. For small \( \beta \), \( \Gamma \beta \) does not
intersect the positive real axis. Therefore a lattice gas with finite-range interaction has no phase transition at high temperature. This result was known but is obtained here with minimum technicality.

The Lee-Yang circle theorem implies that an Ising ferromagnet can have at most one phase transition (at $\xi = 1$). The above proof implies that the zeros of $Q(\xi)$ remain close to the unit circle when a small perturbation (possibly many-body) is added to the original ferromagnetic pair interaction. From this one can deduce the following: An infinite Ising ferromagnet has only one equilibrium state at $\xi = 1$; in particular, the thermodynamic limit of the correlation functions is independent of boundary conditions.\(^6\)

(2) Proof of theorem. — When the $\Lambda_\alpha$ are disjoint, $P$ is just the product of the $P_\alpha$ (with disjoint sets of variables) and the theorem is trivial. To prove the theorem in general we first form the product of the $P_\alpha$ with disjoint sets of variables and then obtain $P$ by successive “contractions.” These contractions (introduced by Asano\(^1\)) are described in the following proposition, from which the theorem is immediately obtained.

Lemma: Let $A$, $B$ be closed subsets of $C$ which do not contain 0. Suppose that the complex polynomial

$$a + bz_1 + cz_2 + dz_1z_2$$

can vanish only when $z_1 \in A$ or $z_2 \in B$. Then

$$a + dz$$

can vanish only when $z \in -AB$.

Since $0 \in A$, $0 \in B$, we have $a \neq 0$. If $d = 0$ there is nothing to prove. If $d \neq 0$, $ad - bc = 0$, we have

$$a + bz_1 + cz_2 + dz_1z_2 = d(z_1 + c/d)(z_2 + a/c)$$

therefore $-c/d \in B$, and $a/c \in B$, and $a + dz$ vanishes only when $z = -a/d \in -AB$.

Let now $d \neq 0$, $ad - bc \neq 0$, and write

$$\psi(z) = -a + bz)/(c + d), \quad \psi(z) = a/dz,$$

where $\psi$, $\psi$ are now considered as mappings of the Riemann sphere (add a point at infinity to $C$, $A_1$ $B$). If we write $\omega = \psi^{-1}$, $z_2 = sz_1$ is equivalent to

$$ab + adz_1 + adz_2 + cdx_1z_2 = 0$$

showing that $\omega^2 = 1$: $\omega$ is an involution. Since $B$ is a proper closed set, $\omega(B)$ cannot be interior to $B$ [otherwise also $\omega^2(B)$ would be interior to $B$]. Thus

$$\omega(B) \cap -B \neq \emptyset$$

where $-B$ is the closure of the complement of $B$.

By assumption $-B \subset \psi(A)$ and since $\psi(A)$ is closed,

$$-B \subset \psi(A).$$

Hence

$$\omega(B) \cap \psi(A) \neq 0,$$

or

$$\psi^{-1}(B) \cap \psi(A) \neq 0,$$

or

$$B \cap \psi(A) \neq 0$$

which proves the lemma.

It is a pleasure to thank F. J. Dyson for constructive criticism and Dr. C. Kaysen for his kind hospitality at the Institute for Advanced Study.

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\(^{1}\)Research sponsored by the Alfred P. Sloan Foundation.


\(^{4}\)In this formula, the product is over the $\alpha$ such that $X \in \Lambda_\alpha$. Notation: $-AB$ is the set of points $-z_1 z_2$ with $z_1 \in A$, $z_2 \in B$.

\(^{5}\)We let $|X|$ be the number of points in $X$.


\(^{7}\)A proof of this result will be published elsewhere.

\(^{8}\)I am indebted to F. J. Dyson for communicating this argument to me, and for his kind permission to reproduce it. Dyson's elegant proof extends to the case where $\psi(z) = -(z + f)/g + h$ and $ah + de = bg + cf$. 

304